# Caringbah High School

Year 12 2024

**Mathematics Extension 2**

HSC Course

Assessment Task 4 – Trial HSC Examination

**General Instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

**Total marks – 100**

**Section I** **10 marks**

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

**Section II** **90 marks**

Attempt Questions 11-16

Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

**Name:** ________________________________ **Class:** ____________

| Marker's Use Only | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Section I** | **Section II** | | | | | | **Total** |
| **Q 1-10** | **Q11** | **Q12** | **Q13** | **Q14** | **Q15** | **Q16** | |
| /10 | /15 | /15 | /15 | /15 | /15 | /15 | /100 |

## Section I

**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10

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**1.** What is the value of $i^{2024}$?

(A) $i$

(B) $-i$

(C) 1

(D) –1

**2.** Points $A$, $B$ and $C$ are represented by the vectors $\underset{\sim}{a} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ m \\ -1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 1 \\ -7 \\ n \end{pmatrix}$ respectively. What values of $m$ and $n$ will ensure that $A$, $B$, and $C$ are collinear?

(A) $m = 1$ and $n = -5$

(B) $m = -1$ and $n = -5$

(C) $m = 1$ and $n = 5$

(D) $m = -1$ and $n = 5$

**3.** An insect is crawling in a spiral down and around a sphere with a radius of 1.
The insect's path begins at the top of the sphere and ends at the bottom of the sphere.
Which of the following parametric equations could define the insect's path?

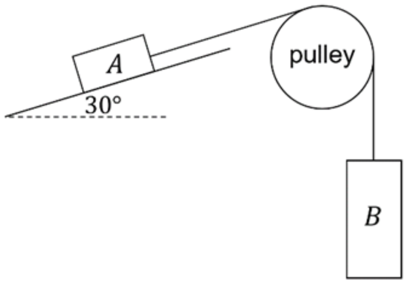
(A) $x = \sin(t)\cos(2t)$, $y = \sin(t)\sin(2t)$, $z = \cos(t)$

(B) $x = \sin(t)\cos(2t)$, $y = \sin(t)\cos(2t)$, $z = \cos(t)$

(C) $x = \cos(t)\cos(2t)$, $y = \sin(t)\cos(2t)$, $z = \cos(t)$

(D) $x = \cos(t)\cos(2t)$, $y = \sin(t)\sin(2t)$, $z = \cos(t)$

**4.** Particle $A$ and particle $B$, of masses $M_A$ and $M_B$ respectively, are connected by a light inextensible string passing over a frictionless pulley as shown.




NOT TO SCALE

Particle $A$ lies on a frictionless surface inclined at 30° to the horizontal, while particle $B$ hangs vertically from the string. The particles are initially at rest, and when they are released, neither particle moves.

Which expression shows the relationship between $M_A$ and $M_B$?

(A) $M_A = M_B$

(B) $M_A = \sqrt{3}M_B$

(C) $M_A = 2M_B$

(D) $M_A = 2\sqrt{3}M_B$

**5.** A particle moves in simple harmonic motion with velocity $v$ ms$^{-1}$.
The motion is described by the equation $v^2 = 25 - 5x^2$, where $x$ is the particle's displacement from the origin. What is the period of the particle's motion?

(A) –5

(B) $\dfrac{2\pi}{25}$

(C) $\dfrac{2\pi}{\sqrt{5}}$

(D) 5

**6.** Consider the statement:

$$\forall\ n \in \mathbb{Z}, \text{ if } n^2 \text{ is odd then } n \text{ is odd}$$

Which of the following is the contrapositive of this statement?

(A) $\forall\ n \in \mathbb{Z}$, if $n$ is not odd then $n^2$ is not odd

(B) $\exists\ n \in \mathbb{Z}$, if $n$ is not odd then $n^2$ is not odd

(C) $\forall\ n \in \mathbb{Z}$, if $n$ is not odd then $n^2$ is odd

(D) $\exists\ n \in \mathbb{Z}$, if $n$ is not odd then $n^2$ is odd

**7.** If $y = re^{i\theta}$, which of the following statements is FALSE?

(A) $\dfrac{r^2}{y} = \overline{y}$

(B) $|y^2| = r^2$

(C) $\text{Arg}(2y) = 2\theta$

(D) $iy = re^{i\left(\theta + \frac{\pi}{2}\right)}$

**8.** It is given that $f(a) = -2, f(b) = 8, g(a) = 1$ and $g(b) = 2$.

What is the value of $\displaystyle\int_a^b \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\, dx$?

(A) 2

(B) 6

(C) –6

(D) –18

**9.** What is the value of $\displaystyle\int_0^2 \frac{2x+2}{x^2+4}\,dx$?

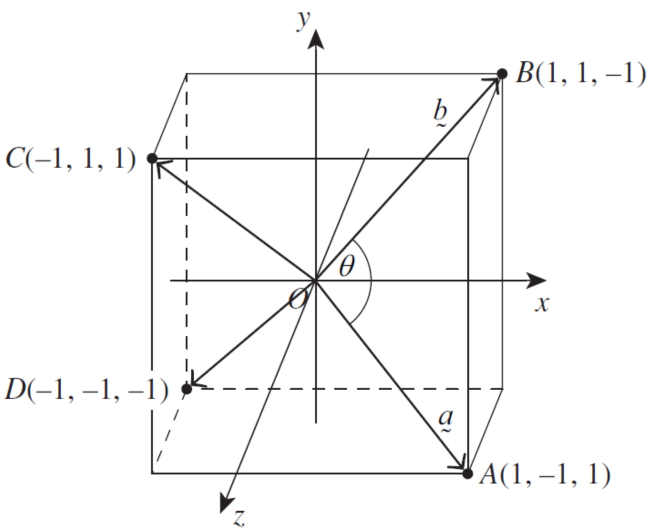
(A) $\ln 2 + \dfrac{\pi}{2}$

(B) $\ln 2 + \dfrac{\pi}{4}$

(C) $\ln 2$

(D) $2 \ln 2$

**10.** The molecular structure of methane can be modelled by a tetrahedron, as shown in the diagram below.




The bond angle, $\theta$, is the angle between any two carbon atoms ($A$, $B$, $C$, $D$). What is the size of the bond angle, correct to one decimal place?

(A) 54.7°

(B) 70.5°

(C) 109.5°

(D) 179.0°

**End of Section I**

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours 45 minutes for this section**

Answer each question in a SEPARATE writing booklet.
Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

**(a)** Consider $z = 2 - 5i$ and $w = -3 - i$. Find simplified expressions for:

**(i)** $z - w$ **1**

**(ii)** $z\overline{z}$ **1**

**(iii)** $\dfrac{z}{w}$ **2**

**(b)** Find $\displaystyle\int \frac{x^6 + 3x^2 - 1}{x^3 + 1}\, dx$ **2**

**(c)** Prove algebraically that the difference between two consecutive square numbers is odd. **2**

**(d)** Given $z = 2e^{i\frac{2\pi}{3}}$, express $z^4$ in the form $a + ib$. **2**

**(e)** Let $\ell_1$ be the straight line segment with equation $\underset{\sim}{r}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $-3 \le \lambda \le 1$. **3**

The straight-line segment $\ell_2$ is perpendicular to $\ell_1$, and the ends of $\ell_2$ are also its $x$- and $y$-intercepts.

Given that the endpoint of the position vector $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ lies on both $\ell_1$ and $\ell_2$,

find the vector equation of $\ell_2$ in the form $\underset{\sim}{r} = \underset{\sim}{a} + \mu \underset{\sim}{b}$, including any restrictions on the value of $\mu$.

**(f)** Find $\displaystyle\int \frac{dx}{\sqrt{4 + 2x - x^2}}$ **2**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

**(a)** Consider the vector equations of the lines

$$r(t) = \begin{pmatrix} -2 + 2\lambda \\ 1 + \lambda \\ 4 - 2\lambda \end{pmatrix} \text{ and } w(s) = \begin{pmatrix} 1 + \mu \\ 3 + 2\mu \\ 2 - 2\mu \end{pmatrix}$$

**(i)** Determine whether the lines intersect. **2**

**(ii)** Hence, or otherwise, determine whether the lines are parallel, perpendicular or skew. Justify your answer. **2**

**(b)** Consider $z = \cos\theta + i\sin\theta$.

**(i)** Show that $z^n + z^{-n} = 2\cos n\theta$ **1**

**(ii)** Hence, solve $3(z^2 + z^{-2}) - (z + z^{-1}) + 2 = 0$ . **3**

**(c)** A particle moves along the $x$-axis according to the equation $\ddot{x} = 1 - x$. Initially, the particle is at rest at $x = 0$. **3**

Find the displacement of the particle as a function of time.

**(d)** **(i)** For a, $b \geq 0$, prove that $a + b \geq 2\sqrt{ab}$. **2**

**(ii)** It is known that the graphs of $y = \sec^2 x$ and $y = 2|\tan x|$ lie on or above the $x$-axis. (Do NOT prove this.) **2**

Using the result from (i), or otherwise, prove that the graph of $y = \sec^2 x$ lies on or above the graph of $y = 2|\tan x|$.

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

**(a)** Find the vector equation of the sphere $x^2 + y^2 + z^2 + 2x - 14y + 25 = 0$. **2**

**(b)** Using the substitution $u = \sin x + 1$, or otherwise, find $\displaystyle\int \frac{\cos x}{\sin^2 x + 2\sin x + 5}\,dx$ **2**

**(c)** Given the complex number $z = x + iy$, sketch and shade the subset on the Argand diagram such that $|z| \leq \sqrt{10}$ and $\text{Im}(z^2) \geq 6$. **3**

**(d)** Prove by contradiction that if $m^2 - 4n = 3$, then $m$ and $n$ cannot both be integers. **3**

**(e)** A particle is travelling in a straight line. Its displacement, $x$ cm, from $O$ at a given time $t$ seconds after the start of motion is given by $x = 1 + \sin^2 t$.

**(i)** Prove that the particle is moving in simple harmonic motion. **2**

**(ii)** Find the centre of motion. **1**

**(iii)** Find the total distance travelled by the particle in the first $\frac{3\pi}{2}$ seconds. **2**

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

**(a)** A stationary object suspended in a vacuum is moved by two forces $F_A$ and $F_B$ where the magnitude of $F_A$ is twice the magnitude of $F_B$. **2**

If the directions of $F_A$ and $F_B$ are $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ respectively, calculate the direction in which the object moves. Give your answer in vector form.

**(b)** Evaluate $\displaystyle\int_0^1 \frac{dx}{1+e^x}$ **3**

**(c)** It is given that the complex number $z \neq -1$ lies on the unit circle in the Argand diagram and $w = \dfrac{1}{z+1}$. Show that $\mathrm{Re}(w) = \dfrac{1}{2}$. **2**

**(d)** Use integration by parts to evaluate $\displaystyle\int_0^1 \sin^{-1} x \, dx$ **3**

**(e)** **(i)** Show that for $k \geq 0$, $2k + 3 > 2\sqrt{(k+2)(k+1)}$ **1**

**(ii)** By decomposing $2k+3$, show that for $k \geq 0$, **2**

$$\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2} - \sqrt{k+1}\right)$$

**(iii)** Hence, or otherwise, show that for $n \geq 1$, **2**

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$$

**End of Question 14**

**Question 15** (15 marks) Use a SEPARATE writing booklet.

**(a)** **(i)** Using the substitution $u = a - x$ or otherwise, prove that **1**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

**(ii)** Hence, evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ **3**

**(b)** Prove that $m!n! < (m+n)!$ for positive integers $m$ and $n$. **2**

**(c)** The acceleration $a$ ms$^{-2}$ of a particle $P$ moving in a straight line is given by

$$a = 4x(x^2 - 3)$$

where $x$ metres is the displacement of the particle to the right of the origin. Initially, the particle is at the origin and is moving with a velocity of 10 ms$^{-1}$.

**(i)** Show that the velocity $v$ ms$^{-1}$ of the particle is given by **2**

$$v^2 = 2x^4 - 12x^2 + 100$$

**(ii)** Describe the motion of the particle, justifying your answer with relevant mathematical reasoning. **2**

**(d)** **(i)** It is known that $|\underset{\sim}{a}| = 1$, $|\underset{\sim}{b}| = 2$, $|\underset{\sim}{c}| = 4$, $\underset{\sim}{a}\cdot\underset{\sim}{b} = -2$ and $\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = -4$. Using dot products, show that $|\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}|^2 = 9$. **2**

**(ii)** By considering the angles between the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or otherwise, show that $|\underset{\sim}{a}-\underset{\sim}{b}+\underset{\sim}{c}| = 7$. **3**

**End of Question 15**

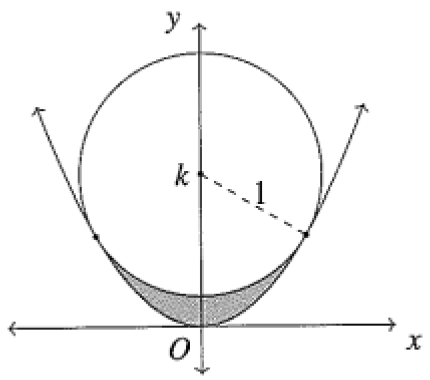
**Question 16** (15 marks) Use a SEPARATE writing booklet.

**(a)** Find the greatest value of the moduli of the complex number $z$ that satisfies the equation $\left|z-\dfrac{4}{z}\right|=2$. **2**

**(b)** Find $\displaystyle\int \cos\sqrt{x}\,dx$ **3**

**(c)** The diagram shows a circle of radius 1, with its centre on the $y$-axis, that is tangent to the parabola $y=x^2$ at two distinct points. **5**

The circle has centre $(0, k)$ for some value $k>0$ and therefore has equation $x^2+(y-k)^2=1$. (Do NOT prove this.)




By showing the $x$-values of the tangent points are $x=\pm\dfrac{\sqrt{3}}{2}$, find the exact area between the circle and the parabola.

**Question 16 continues on page 12**

**(d)** Let $f(x) = 1 + \frac{1}{x}$ and $\varphi = \frac{1 + \sqrt{5}}{2}$.

Define $f \circ f(x)$ to be the composition of $f(x)$ with itself twice, that is $f \circ f(x) = f(f(x))$.

Define $f^n(x)$ to be the function $f(x)$ composed with itself *n* times, for integers $n \geq 0$, that is,

$$f^n(x) = \begin{cases} 1 & n = 0 \\ \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}(x) & n \geq 1 \end{cases}$$

**(i)** Show that $1 + \frac{1}{\varphi} = \varphi$ and $1 + \frac{1}{1 - \varphi} = 1 - \varphi$. **1**

**(ii)** Show, using mathematical induction, that for all integers $n \geq 0$, **3**

$$f^n(1) = \frac{\varphi^{n+2} - (1 - \varphi)^{n+2}}{\varphi^{n+1} - (1 - \varphi)^{n+1}}$$

**(iii)** Hence, or otherwise, find the value of the infinite composition $\lim_{n \to \infty} f^n(1)$. **1**

**End of Examination**

# 2024 Ext 2 AT4 Trial HSC Solutions

## Section 1

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| C | D | A | C | C | A | C | B | B | C |

1. $i^{2024} = (i^4)^{506}$

$= 1^{506}$

$= 1$ $\quad\quad\therefore C$

2. For A, B, C collinear, $\vec{AC} = k\vec{AB}$ (k constant)

$$\vec{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 4 \\ m \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ m-1 \\ 2 \end{pmatrix}$$

$$\vec{AC} = \underset{\sim}{c} - \underset{\sim}{a} = \begin{pmatrix} 1 \\ -7 \\ n \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -8 \\ n+3 \end{pmatrix}$$

Now, $\begin{pmatrix} -4 \\ -6 \\ n+3 \end{pmatrix} = k\begin{pmatrix} -1 \\ m-1 \\ 2 \end{pmatrix} \Rightarrow k = 4$

$-8 = 4m - 4$ and $n + 3 = 8$

$-4 = 4m$ $\quad\quad n = 5$

$m = -1$ $\quad\quad\therefore D$

3. Test $x^2 + y^2 + z^2 = 1$

A: $x^2 + y^2 + z^2 = (\sin t \cos 2t)^2 + (\sin t \sin 2t)^2 + (\cos t)^2$

$= \sin^2 t \cos^2 2t + \sin^2 t \sin^2 2t + \cos^2 t$

$= \sin^2 t\,(\cos^2 2t + \sin^2 2t) + \cos^2 t$

$= \sin^2 t + \cos^2 t$, since $\sin^2\theta + \cos^2\theta = 1$

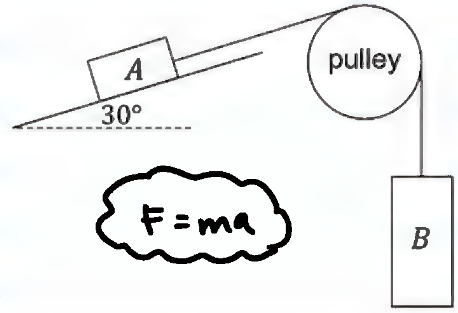
$= \sin^2 t\,(\cos^2 2t + \sin^2 2t) + \cos^2 t$

$= \sin^2 t + \cos^2 t$, since $\sin^2\theta + \cos^2\theta = 1$

$= 1$ as required

$\therefore$ A

4. Equating vertical components




$$M_A \times \sin 30° = M_B$$

$$M_A \times \frac{1}{2} = M_B$$

$F = ma$

$$M_A = 2M_B \qquad \therefore C$$

5. $v^2 = 25 - 5x^2$

$v^2 = -5(x^2 - 5)$

$v^2 = -n^2(x^2 - a^2)$

$n = \sqrt{5}, \ a = \sqrt{5}$

Period $= \frac{2\pi}{n} = \frac{2\pi}{\sqrt{5}}$ $\qquad \therefore C$

6. Contrapositive of $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$

So either A or B.

Original statement applies to ALL $n$, $\therefore A$

7. $y = re^{i\theta}$

Testing A: $\frac{r^2}{y} = \frac{r^2}{re^{i\theta}}$

$= re^{-i\theta}$

$= \bar{y}$ ✓ True

Testing B: $|y^2| = |(re^{i\theta})^2|$

$= |r^2 e^{2i\theta}|$

$= r^2$ ✓ True

Testing C: $\text{Arg}(2y) = \text{Arg}(2re^{i\theta})$

$= \theta$

$\neq 2\theta$ ✗ False $\qquad \therefore C$

8. $\displaystyle\int_a^b \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}\,dx = \int_a^b \left(\frac{f(x)}{g(x)}\right)' dx$

$\displaystyle = \left[\frac{f(x)}{g(x)}\right]_a^b$

$\displaystyle = \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)}$

$\displaystyle = \frac{8}{2} - \frac{-2}{1}$

$= 6$ $\quad\therefore B$

9. $\displaystyle\int_0^2 \frac{2x+2}{x^2+4}\,dx = \int_0^2 \frac{2x}{x^2+4} + \frac{2}{x^2+4}\,dx$

$\displaystyle = \int_0^2 \frac{(x^2+4)'}{x^2+4} + \frac{2}{x^2+4}\,dx$

$\displaystyle = \left[\ln(x^2+4)\right]_0^2 + \frac{2}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \ln 8 - \ln 4 + \left(\tan^{-1}(1) - \tan^{-1}(0)\right)$

$= \ln(8/4) + (\pi/4 - 0)$

$= \ln 2 + \pi/4$ $\quad\therefore B$

10. $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$

$\displaystyle \cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$ $\qquad \underset{\sim}{u} = (1, 1, -1)$

$\qquad \underset{\sim}{v} = (1, -1, 1)$

$\displaystyle = \frac{1 \times 1 + 1 \times -1 + -1 \times 1}{\sqrt{1^2+1^2+(-1)^2} \times \sqrt{1^2+(-1)^2+1^2}}$

$\displaystyle = \frac{-1}{3}$

$\theta = \cos^{-1}(-1/3)$

3

$\theta = \cos^{-1}(-1/3)$

$= 109.5^\circ$ (1dp)          $\therefore C$

Section 2

## Question 11

(a) $z = 2-5i,\ w = -3-i$

(i) $z - w = 2-5i-(-3-i)$
$= 2-5i+3+i$
$= 5-4i$ ✓

(ii) $z\bar{z} = (2-5i)(2+5i)$
$= 4+25$
$= 29$ ✓

(iii) $\frac{z}{w} = \frac{2-5i}{-3-i} \times \frac{-3+i}{-3+i}$
$= \frac{-6+2i+15i+5}{(-3)^2+1}$ ✓
$= \frac{-1+17i}{10}$ ✓

(b) $\int \frac{x^6+3x^2-1}{x^3+1}\,dx = \int \frac{(x^3)^2-1+3x^2}{x^3+1}\,dx$
$= \int \frac{(x^3+1)(x^3-1)}{x^3+1} + \frac{3x^2}{x^3+1}\,dx$ ✓
$= \int x^3-1+\frac{(x^3+1)'}{x^3+1}\,dx$
$= \frac{x^4}{4} - x + \ln|x^3+1| + C$ ✓

(c) Let two consecutive square numbers be $k^2, (k+1)^2, k \in \mathbb{Z}$

Then, $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2$ ✓

$= 2k+1$, which is odd ✓

$\therefore$ Difference between two consecutive square numbers is odd

(d) $z = 2e^{i\frac{2\pi}{3}}$

$z^4 = \left(2e^{i\frac{2\pi}{3}}\right)^4$

$= 2^4 e^{i\frac{8\pi}{3}}$

$= 16 e^{i\frac{2\pi}{3}}$ ✓

$= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -8 + 8\sqrt{3}i$ ✓

(e) $l_1: \quad \underset{\sim}{r} = \begin{pmatrix}6\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\end{pmatrix}, \quad -3 \le \lambda \le 1$

$l_1 \perp l_2$ and meet at $\begin{pmatrix}2\\2\end{pmatrix}$

$l_2: \quad \underset{\sim}{r} = \begin{pmatrix}2\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-2\end{pmatrix}$ ✓ Direction vector

When $x=0$, $2+\mu=0 \rightarrow \mu=-2$

when $y=0$, $2-2\mu=0 \rightarrow \mu=1$

$\therefore l_2: \quad \underset{\sim}{r} = \begin{pmatrix}2\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-2\end{pmatrix}, \quad -2 \le \mu \le 1$ ✓ Equation ✓ $\mu$ restriction

Alternative: $l_2 = \begin{pmatrix}2\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\2\end{pmatrix}, \quad -1 \le \mu \le 2$

(f) $\int \frac{dx}{\sqrt{4+2x-x^2}} = \int \frac{dx}{\sqrt{5-x^2+2x-1}}$

$= \int \frac{dx}{\quad}$ ✓

$$\int \sqrt{4+2x-x^2} \qquad \int \sqrt{5-x^2+2x-1}$$

$$= \int \frac{dx}{\sqrt{5-(x-1)^2}} \quad \checkmark$$

$$= \sin^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C \quad \checkmark$$

## Question 12

(a) $r(t) = \begin{pmatrix} -2+2\lambda \\ 1+\lambda \\ 4-2\lambda \end{pmatrix}$ and $w(s) = \begin{pmatrix} 1+\mu \\ 3+2\mu \\ 2-2\mu \end{pmatrix}$

(i) For any points of intersection, $r(t) = w(s)$

$$\begin{cases} -2+2\lambda = 1+\mu & \text{—— } ① \\ 1+\lambda = 3+2\mu & \text{—— } ② \\ 4-2\lambda = 2-2\mu & \text{—— } ③ \end{cases}$$

From ①,

$\lambda = 2+2\mu$ —— ④

Sub ④ in ①

$-2+2(2+2\mu) = 1+\mu$

$-2+4+4\mu = 1+\mu$

$3\mu = -1$

$\mu = -1/3$ ✓

Sub $\mu = -1/3$ in ④

$\lambda = 2+2(-1/3)$

$\lambda = 4/3$

Test $\lambda = 4/3$, $\mu = -1/3$ in ③

$4-2(4/3) = 2-2(-1/3)$

$4/3 = 8/3$, which is FALSE! ✓

$\therefore$ No points of intersection.

(ii) Direction vectors of $r(t)$ and $w(s)$ are $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Since $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ for $k \in \mathbb{Z}$, they are not parallel. ✓

Also, $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 2+2+4 = 8 \neq 0$, they are not perpendicular.

Also, $\begin{pmatrix}2\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix} = 2+2+4 = 8 \neq 0$, they are not perpendicular. ✓

$\therefore$ The lines are <u>SKEW</u>.

(b) $z = \cos\theta + i\sin\theta$

(i) RTP: $z^n + z^{-n} = 2\cos n\theta$

$LHS = z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ ✓

$= 2\cos n\theta$

$= RHS$

{ $\sin x$ is even
{ $\cos x$ is odd

(ii) $3(z^2 + z^{-2}) - (z + z^{-1}) + 2 = 0$

$3(2\cos 2\theta) - 2\cos\theta + 2 = 0$

$6(2\cos^2\theta - 1) - 2\cos\theta + 2 = 0$

$12\cos^2\theta - 6 - 2\cos\theta + 2 = 0$

$12\cos^2\theta - 2\cos\theta - 4 = 0$

$6\cos^2\theta - \cos\theta - 2 = 0$ ✓

$(3\cos\theta - 2)(2\cos\theta + 1) = 0$

$\cos\theta = \frac{2}{3}$ $\qquad$ $\cos\theta = -\frac{1}{2}$ ✓

$\sin\theta = \pm\frac{\sqrt{5}}{3}$ $\qquad$ $\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore z = \frac{2 \pm i\sqrt{5}}{3}, \frac{-1 \pm i\sqrt{3}}{2}$ ✓

(c) $\ddot{x} = 1 - x = -(x - 1)$ $\qquad$ $\ddot{x} = -n^2(x - c)$

$\therefore$ SHM, $n = 1$, $c = 1$ ✓

At $t = 0$, $x = 0$, $v = 0$ (one extremity)

$\therefore a = 1$, no phase shift ✓

Let $x = -a\cos(nt) + c$

$\therefore x = -\cos t + 1$ ✓

(d) (i) Consider $(\sqrt{a}-\sqrt{b})^2 \geq 0$ for $a, b \geq 0$ ✓

$$a - 2\sqrt{ab} + b \geq 0$$

$$\therefore a + b \geq 2\sqrt{ab}$$ ✓

(ii) From (i), let $a=1$, $b=\tan^2 x$ ✓

$$1+\tan^2 x \geq 2\sqrt{1 \times \tan^2 x}$$

$$\sec^2 x \geq 2|\tan x| \geq 0$$ ✓

$\therefore$ Graph of $y=\sec^2 x$ lies on or above $y=2|\tan x|$

## Question 13

(a) $x^2+y^2+z^2+2x-14y+25=0$

$(x^2+2x+1)+(y^2-14y+49)+z^2=-25+1+49$

$(x+1)^2+(y-7)^2+z^2=25$ ✓

$\therefore \left| \underset{\sim}{x} - \begin{pmatrix} -1 \\ 7 \\ 0 \end{pmatrix} \right| = 5$ ✓

(b) $\displaystyle\int \frac{\cos x}{\sin^2 x+2\sin x+5}\,dx$

$u=\sin x+1 \rightarrow \sin x = u-1$

$du = \cos x\, dx$

$= \displaystyle\int \frac{1}{(u-1)^2+2(u-1)+5}\,du$

$= \displaystyle\int \frac{1}{u^2-2u+1+2u-2+5}\,du$

$= \displaystyle\int \frac{1}{u^2+4}\,du$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)+c$

$= \frac{1}{2}\tan^{-1}\left(\frac{\sin x+1}{2}\right)+c$ ✓

(c) $|z|\leq\sqrt{10}$ <u>AND</u> $\text{Im}(z^2)\geq 6$

$z=x+iy$

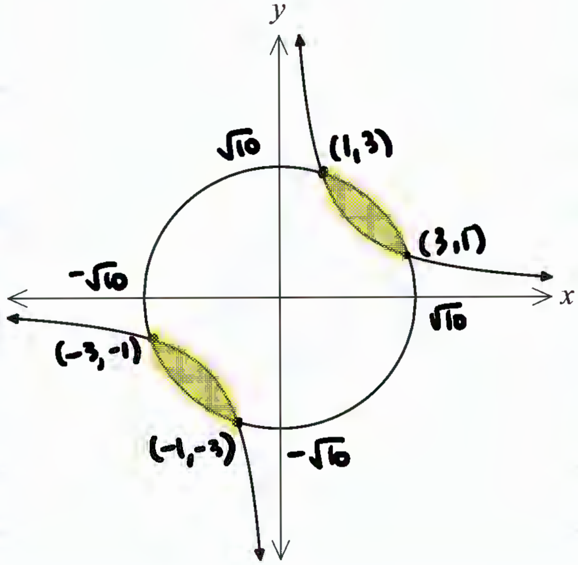
$z^2=(x+iy)^2=x^2-y^2+2ixy$

$x^2+y^2\leq 10$

$2xy\geq 6$

$xy\geq 3$

✓ both




✓ Graphs

✓ Region

(d) Suppose that $m^2 - 4n = 3$ such that $m, n \in \mathbb{Z}$

Since 3 is odd, then $m^2 - 4n$ must be odd

Let $m^2 - 4n = 2k+1,\ k \in \mathbb{Z}$

$$m^2 = 2k + 4n + 1$$

$$= 2(k+2n)+1, \text{ which is odd}$$

$\therefore m^2$ is odd, so $m$ must be odd

Let $m = 2j+1,\ j \in \mathbb{Z}$

Then $(2j+1)^2 + 4n = 3$

$$4j^2 + 4j + 1 + 4n = 3$$

$$4(j^2 + j + n) = 2$$

This is a contradiction since LHS is a multiple of 4 and RHS is not.

$\therefore$ By contradiction, if $m^2 - 4n = 3$, $m$ and $n$ cannot both be integers.

(e) $x = 1 + \sin^2 t \rightarrow \sin^2 t = x - 1$

(i) $\dot{x} = 2\sin t \cos t$

$$\ddot{x} = 2(\cos t \cos t + \sin t . -\sin t)$$

$$= 2(\cos^2 t - \sin^2 t)$$

$$= 2[(1 - \sin^2 t) - \sin^2 t]$$

$$= 2(1 - 2\sin^2 t)$$

$$= -2(2(x-1) - 1)$$

$$= -4((x-1) - \tfrac{1}{2})$$

$$= -4(x - 3/2) \qquad \ddot{x} = -n^2(x-c)$$

$\therefore$ SHM, $n = 2$, $c = 3/2$

(ii) COM at $x = 3/2$

(iii) Period $= 2\pi/n = \pi$ seconds

Amplitude: $\dot{x} = 2\sin t \cos t = 0$ at endpoints of motion

$$t = 0, \tfrac{\pi}{2}, \pi, \ldots$$

$$x = 1, 2, 1, \ldots$$

$$\therefore 1 \le x \le 2,\ a = 1/2$$

$x = 1, 2, 1, \ldots$

$\therefore 1 \le x \le 2, \quad a = \frac{1}{2}$ ✓

In the first $\frac{3\pi}{2}$ seconds, 1.5 oscillations travelled

$\therefore$ travelled 3m in first $\frac{3\pi}{2}$ seconds ✓

# Question 14

(a)
$$\hat{F}_A = \frac{1}{\sqrt{(-1)^2+2^2+1^2}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

$$\hat{F}_B = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Direction $= 2\hat{F}_A + \hat{F}_B$

$$= \frac{2}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{3}}\left[\frac{2}{\sqrt{2}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}\right]$$
$$= \frac{1}{\sqrt{3}}\left[\sqrt{2}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}\right]$$
$$= \frac{1}{\sqrt{3}}\begin{pmatrix}1-\sqrt{2}\\2\sqrt{2}+1\\\sqrt{2}+1\end{pmatrix}$$

(b)
$$\int_0^1 \frac{dx}{1+e^x} = \int_0^1 \frac{1+e^x-e^x}{1+e^x}\,dx$$
$$= \int_0^1 1 - \frac{e^x}{1+e^x}\,dx$$
$$= \Big[x - \ln(1+e^x)\Big]_0^1$$
$$= 1 - \ln(1+e^1) - (0 - \ln(1+e^0))$$
$$= 1 - \ln(1+e) + \ln 2$$

$$= 1 - \ln(1+e^1) - (0 - \ln(1+e^0))$$

$$= 1 - \ln(1+e) + \ln 2$$

$$= 1 + \ln\left(\frac{2}{1+e}\right) \text{ or } \ln\left(\frac{2e}{1+e}\right)$$

(c) $w = \frac{1}{z+1}, \quad z \neq -1$

Let $z = x + iy, \qquad x^2 + y^2 = 1$

$$w = \frac{1}{x+iy+1}$$

$$= \frac{1}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x+1)-iy}{(x+1)^2+y^2}$$

$$Re(w) = \frac{x+1}{(x+1)^2+y^2}$$

$$= \frac{x+1}{x^2+2x+1+y^2}$$

$$= \frac{x+1}{2x+1+1} \qquad \text{since } x^2+y^2=1$$

$$= \frac{x+1}{2(x+1)}$$

$= \frac{1}{2}$ as required

(d) $\int_0^1 \sin^{-1} x \, dx$

$u = \sin^{-1} x \qquad v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$$= \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

Let $t = 1 - x^2$

$dt = -2x \, dx$

$\begin{cases} x=0, & t=1 \\ x=1, & t=0 \end{cases}$

$$= 1 \times \sin^{-1}(1) - 0 + \frac{1}{2}\int_1^0 \frac{dt}{\sqrt{t}}$$

$$= \frac{\pi}{2} + \frac{1}{2}\int_1^0 t^{-1/2} dt$$

$$= \frac{\pi}{2} + \frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right]_1^0$$

$$= \frac{\pi}{} + \sqrt{0} - \sqrt{1}$$

$= \frac{\pi}{2} + \sqrt{0} - \sqrt{1}$

$= \frac{\pi}{2} - 1$ ✓

(e) (i) RTP: $2k+3 \geq 2\sqrt{(k+2)(k+1)}, \quad k \geq 0$

$$LHS^2 - RHS^2 = (2k+3)^2 - \left(2\sqrt{(k+2)(k+1)}\right)^2$$
$$= 4k^2 + 12k + 9 - 4(k+2)(k+1)$$
$$= 4k^2 + 12k + 9 - 4(k^2+3k+2)$$
$$= 4k^2 + 12k + 9 - 4k^2 - 12k - 8$$
$$= 1$$
$$> 0$$

$\therefore LHS^2 > RHS^2, \quad LHS > RHS$

(ii)
$$2k+3 > 2\sqrt{(k+2)(k+1)}$$
$$2k+2+1 > 2\sqrt{(k+2)(k+1)}$$
$$2(k+1)+1 > 2\sqrt{(k+2)(k+1)}$$
$$\frac{2(k+1)}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2}$$
$$2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2}$$
$$\therefore \frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2} - \sqrt{k+1}\right)$$

(iii) From (ii), $\frac{1}{\sqrt{0+1}} = 1 > 2(\sqrt{2} - \sqrt{1})$

$$\frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} > 2(\sqrt{3} - \sqrt{2})$$
$$\frac{1}{\sqrt{3}} > 2(\sqrt{4} - \sqrt{3})$$
$$\vdots$$
$$\frac{1}{\sqrt{n-1}} > 2(\sqrt{n} - \sqrt{n-1})$$
$$\frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - \sqrt{n})$$

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} > 2\left[\cancel{\sqrt{2}} - 1 + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \sqrt{4} - \cancel{\sqrt{3}} + \dots + \cancel{\sqrt{n}} - \sqrt{n-1} + \sqrt{n+1} - \cancel{\sqrt{n}}\right]$$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n}}>2\left[\cancel{\sqrt{2}}-1+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\sqrt{4}-\cancel{\sqrt{3}}+\dots+\cancel{\sqrt{n}}-\sqrt{n-1}+\sqrt{n+1}-\cancel{\sqrt{n}}\right]$$

✓

$$\therefore 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$$

## Question 15

(a) (i) $\int_0^a f(x)\,dx$

$u = a - x \rightarrow x = a - u$

$du = -dx$

$x = 0,\ u = a$

$x = a,\ u = 0$

$= \int_a^0 f(a-u) \cdot -du$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$ "Dummy variable" ✓

(ii) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$

$= \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$

$= \int_0^\pi \frac{\pi \sin x}{1+\cos^2 x}\,dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$ ✓

$2\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$

Let $u = \cos x$

$du = -\sin x\,dx$

$x = 0,\ u = 1$

$x = \pi,\ u = -1$

$= \pi \int_1^{-1} \frac{-du}{1+u^2}$

$= \pi \int_{-1}^1 \frac{du}{1+u^2}$

$= \pi \left[\tan^{-1} u\right]_{-1}^1$

$= \pi\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$

$= \pi\left(\frac{\pi}{4} - -\frac{\pi}{4}\right)$

$= \frac{\pi^2}{2}$ ✓

$\therefore \int^\pi x\sin x \quad dx = \frac{\pi^2}{}$

$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

(b) RTP: $m!n! < (m+n)!$ $\quad m, n \in \mathbb{Z}^+$

For $n > 1$,

$1 < n+1$

$2 < n+2$

$3 < n+3$

$\vdots$

$m < n+m$

Multiplying these inequalities,

$1 \times 2 \times 3 \times \ldots \times m < (n+1)(n+2)(n+3)\ldots(n+m)$

$m! < (n+1)(n+2)(n+3)\ldots(n+m)$

$n!m! < n!\,(n+1)(n+2)(n+3)\ldots(n+m)$

$\therefore m!n! < (n+m)!$ as required

(c)(i) $a = 4x(x^2-3)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x^3 - 12x$

$\frac{1}{2}v^2 = \int 4x^3 - 12x \, dx$

$= \frac{4x^4}{4} - \frac{12x^2}{2} + c$

$= x^4 - 6x^2 + c$

At $t=0$, $x=0$, $v=10$

$\frac{1}{2}(10)^2 = 0^4 - 6(0)^2 + c$

$c = 50$

$\frac{1}{2}v^2 = x^4 - 6x^2 + 50$

$\therefore v^2 = 2x^4 - 12x^2 + 100$

(ii) $v^2 = 2(x^4 - 6x^2 + 50)$

$= 2\left[(x^2-3)^2 + 41\right]$

$> 0$ for all real $x$

At $t=0$, $v=10>0$ and $v \neq 0$ (since $v^2 \neq 0$)

$> 0$ for all real $x$

At $t=0$, $v = 10 > 0$ and $v \neq 0$ (since $v^2 \neq 0$)

$\therefore$ The particle continues moving to the right & never stops, speeding up for $x > \sqrt{3}$

(d) (i) $|\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}|^2 = (\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})$

$$= [(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}]\cdot[(\underset{\sim}{a}+\underset{\sim}{b})+\underset{\sim}{c}]$$

$$= (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + (\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + \underset{\sim}{c}\cdot\underset{\sim}{c}$$

$$= \underset{\sim}{a}\cdot\underset{\sim}{a} + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} + 2\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + \underset{\sim}{c}\cdot\underset{\sim}{c}$$

$$= |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 + 2\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + |\underset{\sim}{c}|^2 \quad \checkmark$$

$$= 1^2 + 2(-2) + 2^2 + 2(-4) + 4^2$$

$$= 1 - 4 + 4 - 8 + 16$$

$$= 9 \text{ as required} \quad \checkmark$$

(ii) $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$$-2 = 1 \times 2 \times \cos\theta$$

$$\cos\theta = -1$$

$$\theta = \pi, \quad \therefore \underset{\sim}{b} = -2\underset{\sim}{a}$$

Now, $\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = -4$

$$\underset{\sim}{c}\cdot(\underset{\sim}{a}-2\underset{\sim}{a}) = -4$$

$$-\underset{\sim}{a}\cdot\underset{\sim}{c} = -4$$

$$\underset{\sim}{a}\cdot\underset{\sim}{c} = 4 \quad \checkmark$$

$$\underset{\sim}{a}\cdot\underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{c}|\cos\varphi$$

$$4 = 1 \times 4 \cos\varphi$$

$$\cos\varphi = 1$$

$$\varphi = 0, \quad \therefore \underset{\sim}{c} = 4\underset{\sim}{a} \quad \checkmark$$

So $|\underset{\sim}{a}-\underset{\sim}{b}+\underset{\sim}{c}| = |\underset{\sim}{a} - -2\underset{\sim}{a} + 4\underset{\sim}{a}|$

$$= |7\underset{\sim}{a}|$$

$$= 7|\underset{\sim}{a}|$$

$$= 7 \times 1$$

$$= 7 \text{ as required} \quad \checkmark$$

## Question 16

(a) By the triangle inequality, $|z| - |w| \leq |z - w|$

Let $w = \frac{4}{z}$

$$|z| - \left|\frac{4}{z}\right| \leq \left|z - \frac{4}{z}\right|$$

$$|z| - \frac{4}{|z|} \leq 2$$

$$|z|^2 - 4 \leq 2|z|$$

$$|z|^2 - 2|z| + 1 - 5 \leq 0$$

$$(|z| - 1)^2 - 5 \leq 0$$

$$(|z| - 1)^2 \leq 5$$

$$-\sqrt{5} \leq |z| - 1 \leq \sqrt{5}$$

$$1 - \sqrt{5} \leq |z| \leq 1 + \sqrt{5}$$

$\therefore$ max value of $|z|$ is $1 + \sqrt{5}$ such that $\left|z - \frac{4}{z}\right| = 2$

(b) $\int \cos\sqrt{x}\, dx$

Let $t = \sqrt{x}$

$dt = \frac{1}{2} x^{-1/2}\, dx$

$dt = \frac{1}{2t}\, dx$

$= \int 2t \cos t\, dt$

$= 2 \sin t - \int 2 \sin t\, dt$

$u = 2t \qquad v' = \cos t$

$u' = 2 \qquad v = \sin t$

$= 2t \sin t - 2 \cos t + C$

$= 2\sqrt{x} \sin\sqrt{x} - 2\cos\sqrt{x} + C$

(c) $\begin{cases} x^2 + (y-k)^2 = 1 & \text{— ①} \\ y = x^2 & \text{— ②} \end{cases}$

Sub ② in ①

$$y + (y-k)^2 = 1$$
$$y + y^2 - 2ky + k^2 = 1$$
$$y^2 + (1-2k)y + k^2 - 1 = 0$$
$$\Delta = (1-2k)^2 - 4(1)(k^2-1) = 0 \quad \text{for tangent}$$
$$1 - 4k + 4k^2 - 4k^2 + 4 = 0$$
$$5 = 4k$$
$$k = 5/4 \quad ✓$$

So $y^2 - \frac{3}{2}y + \frac{9}{16} = 0$

$$(y - 3/4)^2 = 0$$
$$y = \frac{3}{4}$$

Sub $y = 3/4$ in ②

$$\frac{3}{4} = x^2 \quad ✓ \text{ Show}$$

$\therefore x = \pm\frac{\sqrt{3}}{2}$ at POIs.

[Q16(c) continued on next page]

$$\text{Area} = \int_a^b \text{Top} - \text{Bottom}\ dx$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{5}{4} - \sqrt{1-x^2} - x^2\ dx$$

$$x^2 + \left(y - \frac{5}{4}\right)^2 = 1$$

$$\left(y - \frac{5}{4}\right)^2 = 1 - x^2$$

$$y - \frac{5}{4} = \pm\sqrt{1-x^2}$$

$$y = \frac{5}{4} - \sqrt{1-x^2}$$

(lower semi-circle)

$$= 2\int_0^{\frac{\sqrt{3}}{2}} \frac{5}{4} - x^2 - \sqrt{1-x^2}\ dx$$

$$= 2\left[\frac{5x}{4} - \frac{x^3}{3}\right]_0^{\frac{\sqrt{3}}{2}} - 2\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\ dx$$

$$= 2\left[\frac{5\sqrt{3}/2}{4} - \frac{(\sqrt{3}/2)^3}{3} - 0\right] - 2\int_0^{\pi/3} \cos\theta\sqrt{1-\sin^2\theta}\ d\theta$$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= 2\left(\frac{5\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}\right) - 2\int_0^{\pi/3} \cos\theta\cos\theta\ d\theta$$

$$= 2\left(\frac{2\sqrt{3}}{8}\right) - 2\int_0^{\pi/3} \frac{1}{2}(\cos 2\theta + 1)\, d\theta$$

$$= \frac{\sqrt{3}}{2} - \left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/3}$$

$$= \frac{\sqrt{3}}{2} - \left(\frac{\sin 2\pi/3}{2} + \frac{\pi}{3} - 0\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}/2}{2} - \frac{\pi}{3}$$

$$= \frac{3\sqrt{3}}{4} - \frac{\pi}{3}\ \text{units}^2$$

(d) $f(x) = 1 + \frac{1}{x}$ , $\varphi = \frac{1+\sqrt{5}}{2}$

$$f^n(x) = \begin{cases} 1 & n = 0 \\ \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}(x) & n \geq 1 \end{cases}$$

(i) $$1 + \frac{1}{\varphi} = 1 + \frac{2}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= 1 + \frac{2(1-\sqrt{5})}{1-5}$$

$$= 1 + \frac{\cancel{2}(1-\sqrt{5})}{-\cancel{4}\,2}$$

$$= \frac{2-1+\sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2}$$

$$= \varphi$$

$$1 + \frac{1}{1-\varphi} = 1 + \frac{1}{1 - \frac{1+\sqrt{5}}{2}}$$

$$= 1 + \frac{1}{\frac{1-\sqrt{5}}{2}}$$

$$= 1 + \frac{2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$

$$= 1 + \frac{2(1+\sqrt{5})}{1-5}$$

$$= 1 + \frac{\cancel{2}(1+\sqrt{5})}{-\cancel{4}\,2}$$

$$= 1 - \frac{1+\sqrt{5}}{2}$$

$$= 1 - \varphi$$

✓ Both

(ii) RTP: $f^n(1) = \dfrac{\varphi^{n+2} - (1-\varphi)^{n+2}}{\varphi^{n+1} - (1-\varphi)^{n+1}}$, $n \geq 0$

① For $n=0$,

$LHS = f^0(1) = 1$

$$RHS = \frac{\varphi^2 - (1-\varphi)^2}{\varphi - (1-\varphi)}$$

$$= \frac{\varphi^2 - 1 + 2\varphi - \varphi^2}{\varphi - 1 + \varphi}$$

$$= \frac{2\varphi - 1}{2\varphi - 1}$$

$$= 1$$

$\therefore$ LHS = RHS, statement is true for $n=0$

② Assume statement is true for $n=k$

i.e.

$$f^k(1) = \frac{\varphi^{k+2} - (1-\varphi)^{k+2}}{\varphi^{k+1} - (1-\varphi)^{k+1}} \qquad k \geq 0$$

Attempt to prove true for $n = k+1$

i.e. $f^{k+1}(1) = \dfrac{\varphi^{k+3} - (1-\varphi)^{k+3}}{\varphi^{k+2} - (1-\varphi)^{k+2}}$

[Q16(d)(ii) continues on next page]

$$\begin{aligned}
LHS &= f^{k+1}(1) \\
&= f\left(f^k(1)\right) \\
&= 1 + \frac{1}{f^k(1)} \\
&= 1 + \frac{\varphi^{k+1} - (1-\varphi)^{k+1}}{\varphi^{k+2} - (1-\varphi)^{k+2}} \qquad \text{from assumption} \\
&= \frac{\varphi^{k+2} - (1-\varphi)^{k+2} + \varphi^{k+1} - (1-\varphi)^{k+1}}{\varphi^{k+2} - (1-\varphi)^{k+2}} \\
&= \frac{\varphi^{k+2}\left(1 + \frac{1}{\varphi}\right) - (1-\varphi)^{k+2}\left(1 + \frac{1}{1-\varphi}\right)}{\varphi^{k+2} - (1-\varphi)^{k+2}} \\
&= \frac{\varphi^{k+2}(\varphi) - (1-\varphi)^{k+2}(1-\varphi)}{\varphi^{k+2} - (1-\varphi)^{k+2}} \qquad \text{from (i)} \\
&= \frac{\varphi^{k+3} - (1-\varphi)^{k+3}}{\varphi^{k+2} - (1-\varphi)^{k+2}} \\
&= RHS
\end{aligned}$$

$\therefore$ statement is true for $n = k+1$ if it is true for $n = k$

③ Statement is true for $n=0$, so it is true for $n=1$
Statement is true for $n=1$, so it is true for $n=2$
$\therefore$ By the principle of mathematical induction, true for $n \geq 0$.

(iii) $$\lim_{n\to\infty} f^n(1) = \lim_{n\to\infty} \frac{\varphi^{n+2} - (1-\varphi)^{n+2}}{\varphi^{n+1} - (1-\varphi)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\varphi^{n+2}}{\varphi^{n+1}} \qquad \text{as } |1-\varphi| < 1, \ \lim_{k\to\infty} (1-\varphi)^k = 0$$

$= \lim \quad \varphi$

$$\lim_{n\to\infty} \varphi^{n+1} \quad \lim_{k\to\infty}(\varphi)$$

$$= \lim_{k\to\infty} \varphi$$

$$= \varphi$$

✓